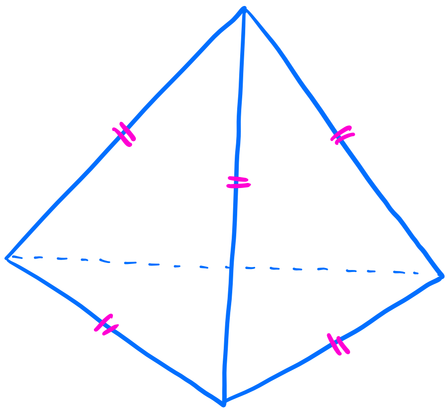
1. A regular tetrahedron has four vertices and any two of those vertices are at the same distance from each other. The four faces of a regular tetrahedron are all equilateral triangles.

(a) (3 points) Sketch a regular tetrahedron.

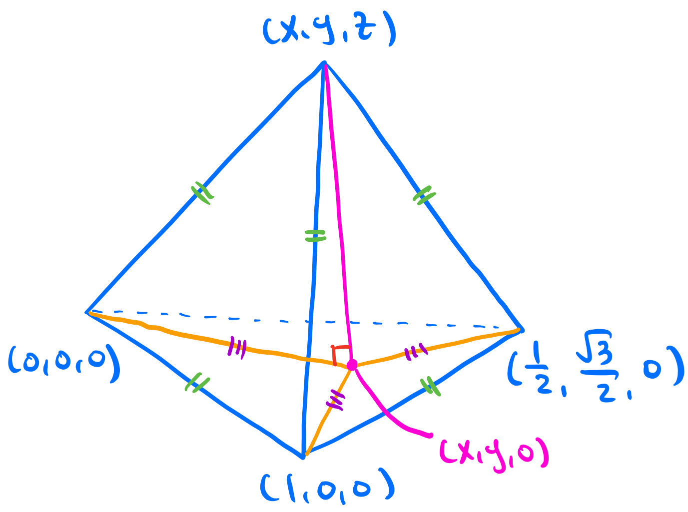


(b) (3 points) What is the angle between any two edges of a regular tetrahedron?

All faces of a regular tetrahedron are equilateral

$\Rightarrow$ All angles on a regular tetrahedron are $\boxed{\frac{\pi}{3}}$

(c) (4 points) The three vertices $(0,0,0)$, $(1,0,0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ all lie in the x-y plane and are the vertices of an equilateral triangle. Find a fourth vertex (x, y, z) that together with the three given vertices forms a tetrahedron.



By symmetry, the projection of (x,y,z) onto the xy-plane must be the center of the base triangle.

$$\Rightarrow \begin{cases} x = \frac{1}{3}(1+0+\frac{1}{2}) = \frac{1}{2} \\ y = \frac{1}{3}(0+0+\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{6}. \end{cases}$$

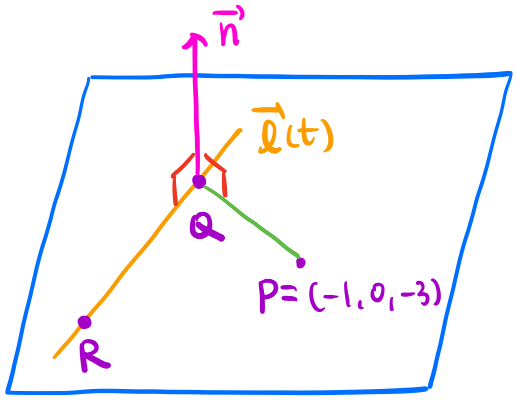
The length of each edge is 1

$$\Rightarrow x^2+y^2+z^2=1 \Rightarrow \left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{6}\right)^2+z^2=1 \Rightarrow z^2=\frac{2}{3} \Rightarrow z=\pm\sqrt{\frac{2}{3}}.$$

The fourth vertex is $\boxed{\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \pm\sqrt{\frac{2}{3}}\right)}$

2. Both parts of this question are about the same plane.

(a) (6 points) Find the equation of the plane containing the line $x = 3t+2$, $y = -2t$, $z = -2t-1$ and the point $(-1, 0, -3)$.



Set $P = (-1, 0, 3)$

Choose two points on the given line.

$t=0$: $Q = (2, 0, -1)$

$t=1$: $R = (5, -2, -3)$

A normal vector $\vec{n}$ is perpendicular to both $\overrightarrow{QP}$ and $\overrightarrow{QR}$.

$\Rightarrow \vec{n} = \overrightarrow{QP} \times \overrightarrow{QR} = (-3, 0, -2) \times (3, -2, -2) = (-4, -12, 6)$

The plane is given by the equation

$$\boxed{-4(x+1) - 12(y-0) + 6(z+3) = 0}$$

Note You can solve this problem by choosing other points on the given line.

(b) (4 points) Find the distance of the origin from the plane of part (a).

The plane equation can be written as

$-4x - 12y + 6z + 14 = 0 \rightsquigarrow 2x + 6y - 3z - 7 = 0$

The distance from the origin $(0,0,0)$ to the plane is

$$\frac{|2\cdot 0 + 6\cdot 0 - 3\cdot 0 + 7|}{\sqrt{2^2 + 6^2 + (-3)^2}} = \boxed{1}$$

3. Consider the space curve $\mathbf{r}(t) = \frac{t}{\sqrt{2}}\mathbf{i} + \frac{t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k}$.

(a) (6 points) Find the integral that gives the length of the curve from $t = -2$ to $t = 2$.

$$\vec{r}(t) = \left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, t^2\right) \Rightarrow \vec{r}\,'(t) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2t\right) \Rightarrow |\vec{r}\,'(t)| = \sqrt{1+4t^2}.$$

The length of the curve from $t=-2$ to $t=2$ is

$$\int_{-2}^{2} |\vec{r}\,'(t)|\,dt = \boxed{\int_{-2}^{2} \sqrt{1+4t^2}\,dt}$$

(b) (2 points) Use the indefinite integral

$$\int \sqrt{1+x^2}\,dx = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\log\left(x+\sqrt{1+x^2}\right)$$

to evaluate the length in part (a).

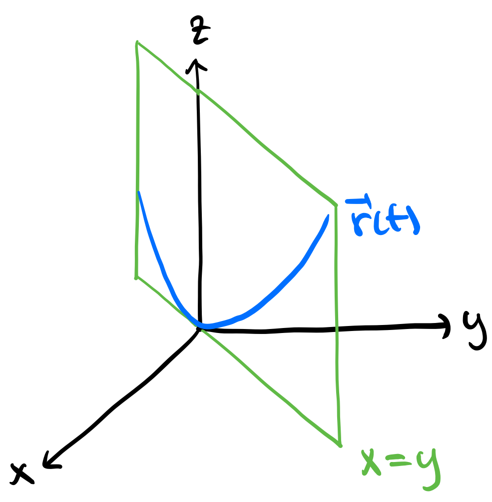
Set $u = 2t \Rightarrow du = 2dt$

$$\int_{-2}^{2} \sqrt{1+4t^2}\,dt = \frac{1}{2}\int_{-4}^{4} \sqrt{1+u^2}\,du = \int_{0}^{4} \sqrt{1+u^2}\,du$$

$$= \frac{u\sqrt{1+u^2}}{2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\Big|_{u=0}^{u=4}$$

$$= \boxed{2\sqrt{17} + \frac{1}{2}\ln\left(4+\sqrt{17}\right)}$$

(c) (2 points) What is the name of the space curve?



$$\vec{r}(t) = \left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, t^2\right)$$

$\Rightarrow x=y$ and $z = x^2+y^2$.

$\Rightarrow$ The curve is the intersection of the plane $x=y$ and the paraboloid $z = x^2+y^2$

$\Rightarrow$ The curve is a $\boxed{\text{parabola}}$

4. The space curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ is a helix. The space curve $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + 2t\,\mathbf{k}$ is the same helix parametrized differently, with t replaced by $2t$.

(a) (2 points) Suppose the position vector of a particle is given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ with t being time. Find its speed $\left|\frac{d\mathbf{r}}{dt}\right|$.

$$\vec{r}(t) = (\cos(t), \sin(t), t) \Rightarrow \vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \boxed{\sqrt{2}}$$

(b) (2 points) Suppose the position vector of a particle is given by $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + 2t\,\mathbf{k}$ with t being time. Find its speed.

$$\vec{r}(t) = (\cos(2t), \sin(2t), 2t) \Rightarrow \vec{r}\,'(t) = (-2\sin(2t), 2\cos(2t), 2)$$

$$|\vec{r}\,'(t)| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 4} = \boxed{2\sqrt{2}}$$

(c) (3 points) Suppose a particle moves on the same helix with initial position $\mathbf{r}(0) = \mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$ with speed equal to $3\sqrt{2}$. Find its position $\mathbf{r}(t)$ as a function of time t.

The particle in (a) moves with speed $\sqrt{2}$.

The particle with speed $3\sqrt{2}$ moves 3 times faster.

$$\Rightarrow \vec{r}(t) = \boxed{(\cos(3t), \sin(3t), 3t)}$$

(d) (3 points) Suppose a particle moves on the same helix with initial position $\mathbf{r}(0) = \mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$ with speed equal to v. Find its position $\mathbf{r}(t)$ as a function of time t.

The particle in (a) moves with speed $\sqrt{2}$.

The particle with speed v moves $\frac{v}{\sqrt{2}}$ times faster.

$$\Rightarrow \vec{r}(t) = \boxed{\left(\cos\left(\frac{v}{\sqrt{2}}t\right), \sin\left(\frac{v}{\sqrt{2}}t\right), \frac{v}{\sqrt{2}}t\right)}$$

5. The equation $x^3 + y^3 + z^3 = 3xyz$ implicitly gives z as a function of x, y and is therefore a surface.

(a) (3 points) Find $\frac{\partial z}{\partial x}$.

$$x^3+y^3+z^3 = 3xyz \rightsquigarrow x^3+y^3+z^3-3xyz = 0.$$

Set $f(x,y,z) = x^3+y^3+z^3-3xyz$.

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{3x^2-3yz}{3z^2-3xy} = \boxed{\frac{yz-x^2}{z^2-xy}}$$

(b) (3 points) Find $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{3y^2-3xz}{3z^2-3xy} = \boxed{\frac{xz-y^2}{z^2-xy}}$$

(c) (4 points) The point $(1, 1, -2)$ lies on the surface. Find the equation of the plane that is tangent to the surface at that point.

The surface is given by $f(x,y,z)=0$

$\Rightarrow$ It is a level surface of $f(x,y,z)$

$\nabla f = (f_x, f_y, f_z) = (3x^2-3yz,\ 3y^2-3xz,\ 3z^2-3xy)$

A normal vector is $\nabla f(1,1,-2) = (9,9,9)$

The tangent plane at $(1,1,-2)$ is given by

$$\boxed{9(x-1)+9(y-1)+9(z+2)=0}$$

<u>Note</u> The answer can be given in many other forms,

such as $x+y+z=0$.

6. Find the partial derivative $\frac{\partial z}{\partial x}$ in both parts.

(a) (5 points) $z = x^2 + y^2$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = \boxed{2x}$$

(b) (5 points) $z = \cos(u + v)$, $u = x^2 - y^2$, $v = x^2 + y^2$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

chain rule

$$= -\sin(u+v) \cdot 2x - \sin(u+v) \cdot 2x$$

$$= \boxed{-4x\sin(u+v)}$$

Note You can always find partial derivatives by direct computation, without using the chain rule.

$$u + v = (x^2 - y^2) + (x^2 + y^2) = 2x^2$$

$$\Rightarrow z = \cos(u+v) = \cos(2x^2)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\sin(2x^2) \cdot 4x$$

However, direct computations are generally much more complicated than computations based on the chain rule.

7. Let ℓ_1 be the line given by $(x, y, z) = (2t, t, 2t)$ and let ℓ_2 be the line given by $(x, y, z) = (-t+3, -2t-3, 2t+3)$.

(a) (1 point) Find the cross-product of the vectors $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $-\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

$$(2, 1, 2) \times (-1, -2, 2) = \boxed{(6, -6, -3)}$$

(b) (3 points) Find a plane that contains ℓ_1 whose normal vector is the same as the cross-product of part (a).

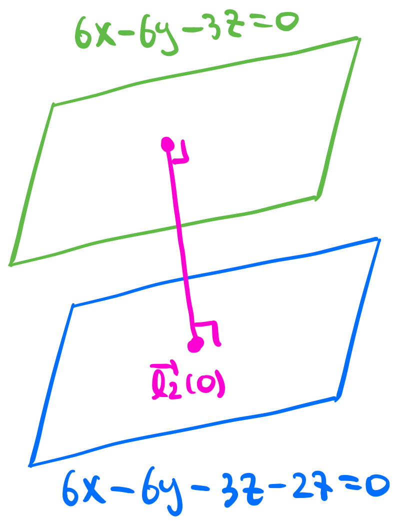
The plane contains $\vec{\ell_1}(0) = (0, 0, 0)$

$\Rightarrow$ The plane is given by $\boxed{6(x-0) - 6(y-0) - 3(z-0) = 0}$

(c) (3 points) Similarly, find a plane that contains ℓ_2 whose normal vector is the same as the cross-product of part (a). Next, find the distance between the two planes.

The plane contains $\vec{\ell_2}(0) = (3, -3, 3)$

$\Rightarrow$ The plane is given by $\boxed{6(x-3) - 6(y+3) - 3(z-3) = 0}$



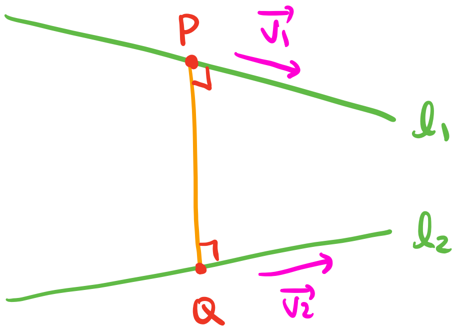
The two planes are parallel as they have the same normal vectors.

$\Rightarrow$ The distance between them equals the distance from $\vec{\ell_2}(0) = (3, -3, 3)$ to the plane $6x - 6y - 3z = 0$.

The distance between the two planes is

$$\frac{|6 \cdot 3 - 6 \cdot (-3) - 3 \cdot 3 + 0|}{\sqrt{6^2 + (-6)^2 + (-3)^2}} = \boxed{3}$$

(d) (3 points) Find the point P on ℓ_1 and the point Q on ℓ_2 such that the distance PQ is minimum and the same as the answer to part (c).



$\overrightarrow{PQ}$ should represent the shortest distance between ℓ_1 and ℓ_2.

$\Rightarrow \overrightarrow{PQ}$ is orthogonal to ℓ_1 and ℓ_2.

P is on $\ell_1 \Rightarrow P = (2t, t, 2t)$

Q is on $\ell_2 \Rightarrow Q = (-u+3, -2u-3, 2u+3)$.

$\overrightarrow{PQ} = (3-2t-u, -3-t-2u, 3-2t+2u)$

Direction vectors of ℓ_1 and ℓ_2 are

$\vec{v_1} = (2, 1, 2)$ and $\vec{v_2} = (-1, -2, 2)$

$\Rightarrow \overrightarrow{PQ} \cdot \vec{v_1} = 0$ and $\overrightarrow{PQ} \cdot \vec{v_2} = 0$

$$\Rightarrow \begin{cases} (3-2t-u, -3-t-2u, 3-2t+2u) \cdot (2, 1, 2) = 0 \\ (3-2t-u, -3-t-2u, 3-2t+2u) \cdot (-1, -2, 2) = 0 \end{cases}$$

$\Rightarrow 9-9t = 0$ and $9+9u = 0$

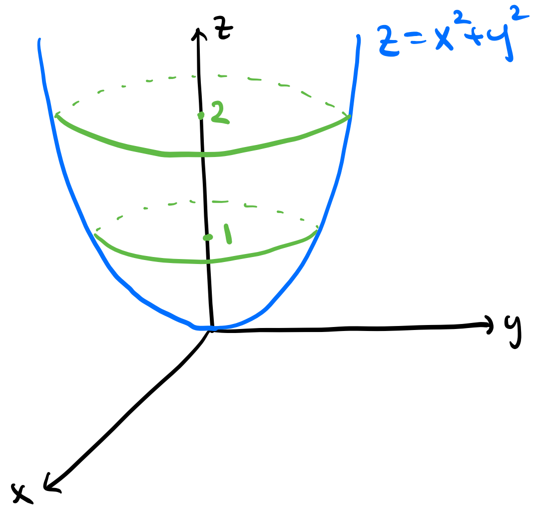
$\Rightarrow t = 1$ and $u = -1$.

$\Rightarrow \boxed{P = (2, 1, 2) \text{ and } Q = (4, -1, 1)}$

<u>Note</u> You can use this method to find the shortest distance between two lines.

8. Consider the paraboloid surface $z = x^2 + y^2$.

(a) (1 point) The base point of the surface is $(0, 0, 0)$ and its axis (of symmetry) is the line $(x, y, z) = (0, 0, t)$. Sketch the surface showing the base point and the axis.



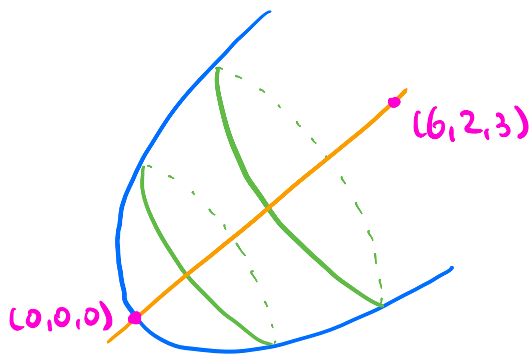
To sketch the surface, we look at cross sections (or traces)

$x=0 \Rightarrow z=y^2$: a parabola

$z=1 \Rightarrow 1=x^2+y^2$: a circle

$z=2 \Rightarrow 2=x^2+y^2$: a circle

(b) (2 points) Now suppose the paraboloid is rotated so that the base point remains the base point but the axis of symmetry is the line $(x, y, z) = (6t, 2t, 3t)$. Sketch the rotated paraboloid showing the base point and the axis.



Note For sketching problems, you should indicate some key features of your graphs, such as

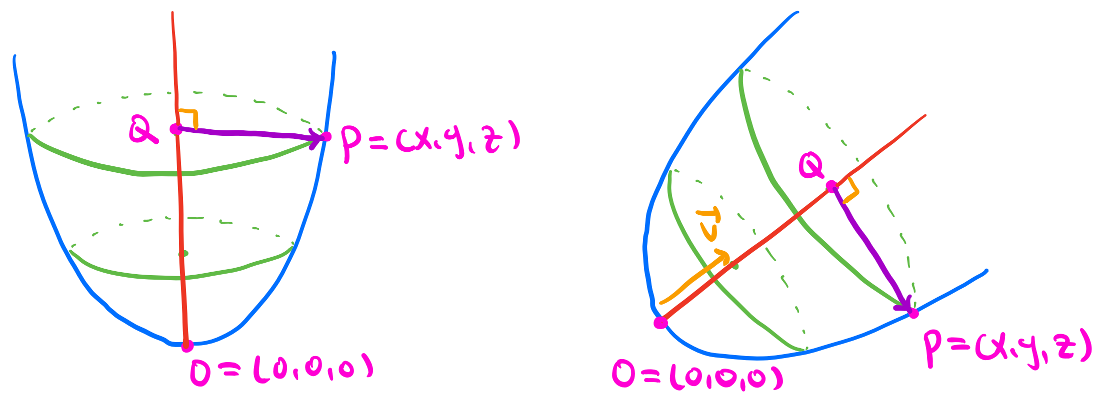
- general shapes of cross sections (or traces)
- line of symmetry (if applicable)
- Some notable points (e.g. intercepts, maxima or minima)

(c) (7 points) Find the equation of the rotated paraboloid of part (b)

Extremely tricky!

Let $P=(x,y,z)$ be a point on the paraboloid.

Set $O=(0,0,0)$ and take Q to be the projection of P onto the line of symmetry.



For the paraboloid of part (a), you get $Q=(0,0,z)$

$\Rightarrow \overrightarrow{OQ}=(0,0,z)$ and $\overrightarrow{QP}=(x,y,0)$

$z=x^2+y^2 \Rightarrow |\overrightarrow{OQ}|=|\overrightarrow{QP}|^2$.

For the rotated paraboloid, we also have $|\overrightarrow{OQ}|=|\overrightarrow{QP}|^2$.

The line of symmetry has a direction vector $\vec{v}=(6,2,3)$.

$$|\overrightarrow{OQ}|=\left|\mathrm{Proj}_{\vec{v}}\overrightarrow{OP}\right|=\left|\frac{\overrightarrow{OP}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}\right|=\frac{\overrightarrow{OP}\cdot\vec{v}}{|\vec{v}|}=\frac{(x,y,z)\cdot(6,2,3)}{\sqrt{6^2+2^2+3^2}}$$

$$=\frac{1}{7}(6x+2y+3z)$$

$$|\overrightarrow{QP}|^2=|\overrightarrow{OP}|^2-|\overrightarrow{OQ}|^2=(x^2+y^2+z^2)-\frac{1}{49}(6x+2y+3z)^2$$

Pythagorean thm.

$$\Rightarrow \boxed{\frac{1}{7}(6x+2y+3z)=(x^2+y^2+z^2)-\frac{1}{49}(6x+2y+3z)^2}$$